Surface and Corner Magnetizations in the Two-Dimensional Ising Model

Christian Kaiser[1] and Ingo Peschel[1]

We study the anisotropic Ising model on a square lattice with free boundary conditions. A simple explicit result is obtained for the surface magnetization of a system which is inhomogeneous near the surface. The corner magnetization is investigated for a 90° corner in a homogeneous anisotropic system. From the numerical results an analytical formula is found.

1. INTRODUCTION

Ordering phenomena near boundaries have been the subject of numerous studies (see ref. 1 for a review). Some particular problems, however, have been investigated only in recent years. One is the case of a system which is inhomogeneous near the boundary and gradually becomes uniform as one moves into the interior. Another is the case of boundaries with edges or corners. Both situations were mainly studied in two dimensions.

The case of the inhomogeneous system was first investigated by Hilhorst and van Leeuwen[2] and later in more detail by Blöte, Hilhorst, and others.[3-5] They considered planar Ising models where some of the couplings approach their bulk values as a power law n^{-y}, where n is the distance from the straight boundary. General renormalization group arguments show that nonuniversal surface exponents can appear if $v \cdot y = 1$, where v is the bulk correlation exponent.[6,7] This was indeed found in the analytical calculations which included surface correlations and the surface magnetization. These calculations, however, used a continuum limit and,

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[1] Fachbereich Physik, Freie Universität Berlin, D-1000 Berlin 33, West Germany.

on the whole, were rather involved. We will show here that it is actually very simple to obtain the surface magnetization m_s in arbitrary layered Ising models. This generalizes an observation made before in the extreme anisotropic (Hamiltonian) limit.[8] It allows us to calculate and discuss m_s for various cases and all temperatures.

The corner problem was first posed by Cardy[9] and studied for the two-dimensional Ising model by Barber *et al.*[10] It turned out that the exponent β_c of the corner magnetization m_c depends on the angle θ at the corner and, in general, also on the anisotropy of the lattice. At the same time it was realized that the angle dependence follows from conformal invariance.[10,11] Analytical calculations so far have been done only for special cases of square lattices.[10,12] In the present paper we present the method and the results for a 90° corner in a general anisotropic square lattice with edges along the directions of the bonds. Although the final steps of the calculation were done numerically, we were able to infer an analytical formula for m_c from the data. This provides the general answer to this problem.

Our treatment is based on a suitable row-to-row transfer matrix and its eigenfunctions. The diagonalization procedure is known in principle but it is important to bring the equations into a convenient form. This is done in the following section, with the emphasis on a clear and simple presentation. We calculate the surface and corner magnetizations in the following sections and draw our conclusions.

2. TRANSFER MATRIX

We consider the square lattice shown in Fig. 1 with free boundary conditions along all edges. The vertical couplings $K_1 = \beta J_1$ are taken to be uniform, while the horizontal ones can vary from column to column, $K_2 = K_2(n)$, corresponding to a layered system. Our tool will be the symmetrized row-to-row transfer matrix

$$W = V_1^{1/2} V_2 V_1^{1/2} \tag{2.1}$$

where[2]

$$V_1 = \exp\left(K_1^* \sum_{n=1}^{N} \sigma_n^z \right) \tag{2.2}$$

$$V_2 = \exp\left[\sum_{n=1}^{N-1} K_2(n)\, \sigma_n^x \sigma_{n+1}^x \right] \tag{2.3}$$

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[2] Note that refs. 13 and 19 have a minus sign in the exponent of V_1. Their results are obtained from ours by letting $K_1^* \to -K_1^*$.
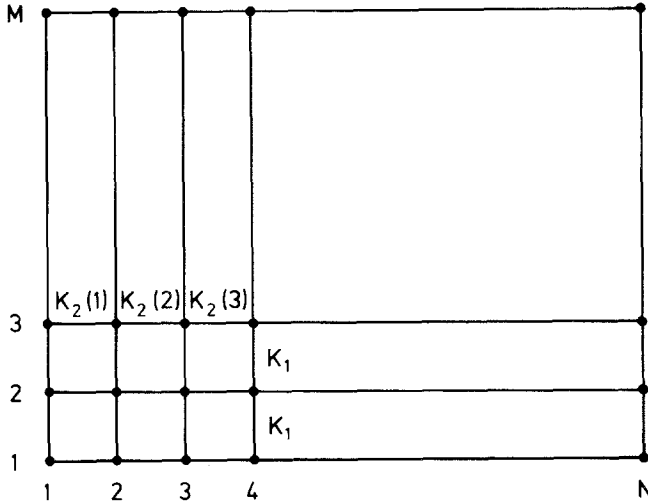
Fig. 1. Geometry of the square lattice and definition of the coupling constants.

The σ_n^α are Pauli matrices and K_1^* denotes the dual coupling of K_1, $\tanh K_1^* = \exp(-2K_1)$. The homogeneous problem was already studied by Abraham,[13] who used Kaufman's spinor technique[14] to diagonalize the related transfer matrix $V = V_2^{1/2} V_1 V_2^{1/2}$. Both W and V have the same eigenvalues but their eigenvectors differ. We work with W because only W allows a simple calculation of the corner magnetization. In the following we use the standard fermion techniques of Lieb, Schultz, and Mattis (LSM)[15] and follow their notation closely.

Introducing Fermi operators c_n, $c_n^\dagger$ via the Jordan-Wigner transformation gives

$$V_1 = \exp\left[K_1^* \sum_{n=1}^{N} (2c_n^\dagger c_n - 1) \right] \tag{2.4}$$

$$V_2 = \exp\left[\sum_{n=1}^{N-1} K_2(n)(c_n^\dagger - c_n)(c_{n+1}^\dagger + c_{n+1}) \right] \tag{2.5}$$

It is then easy to obtain the Heisenberg operators $Wc_n W^{-1}$ from an equation-of-motion technique. Introducing the vector $\mathbf{c} = (c_1, c_2, ..., c_N)$, one has

$$W\mathbf{c}W^{-1} = \mathbf{Mc} + \mathbf{Nc}^\dagger \tag{2.6}$$

$$W\mathbf{c}^\dagger W^{-1} = \hat{\mathbf{M}}\mathbf{c}^\dagger + \mathbf{N}'\mathbf{c} \tag{2.7}$$

The $N \times N$ matrices M, $\hat{M}$, N are tridiagonal and are given explicitly in the appendix. New Fermi operators are then introduced via the Bogoliubov transformation

$$\alpha_k = \sum_{j=1}^{N} \left[g_k(j)\, c_j + h_k(j)\, c_j^\dagger \right] \tag{2.8}$$

such that

$$W = \exp\left[-\sum_k \varepsilon_k(\alpha_k^\dagger \alpha_k - \tfrac{1}{2}) \right] \tag{2.9}$$

is diagonal in the α's. Choosing $\varepsilon_k \geqslant 0$, the vacuum of the α's gives the largest eigenvalue of W. Equations (2.6) and (2.7) determine W only up to a multiplicative constant. It was fixed in (2.9) by demanding that the diagonal form of W lead to the same trace of $\ln W$ as (2.1) (see ref. 15). From (2.9) one has $W\alpha_k W^{-1} = \exp(\varepsilon_k) \cdot \alpha_k$. Inserting here (2.8) and using (2.6) and (2.7) leads to the matrix equations

$$M\mathbf{g}_k + N\mathbf{h}_k = e^{\varepsilon_k}\mathbf{g}_k \tag{2.10}$$

$$\hat{M}\mathbf{h}_k + N'\mathbf{g}_k = e^{\varepsilon_k}\mathbf{h}_k \tag{2.11}$$

with the (column) vector $\mathbf{g}_k = (g_k(1), g_k(2),..., g_k(N))$ and similarly for $\mathbf{h}_k$.

Equations (2.10) and (2.11) determine the canonical transformation (2.8) and the single-fermion eigenvalues ε_k. It is more convenient, however, to introduce the quantities (normalized to one)

$$\boldsymbol{\phi}_k = \mathbf{g}_k + \mathbf{h}_k; \qquad \boldsymbol{\psi}_k = \mathbf{g}_k - \mathbf{h}_k \tag{2.12}$$

By forming proper linear combinations of the equations, one can transform them into a pair of matrix equations for ϕ_k and ψ_k where all matrices are bidiagonal. They read (omitting the index k)

$$\begin{pmatrix} a_1 & b_1 & & & \\ & a_2 & \ddots & & \\ & & \ddots & \ddots & \\ & & & a_{N-1} & b_{N-1} \\ & & & & a_0 \end{pmatrix} \begin{pmatrix} \\ \\ \phi \\ \\ \\ \end{pmatrix} = \begin{pmatrix} c_1 & d_1 & & & \\ & c_2 & \ddots & & \\ & & \ddots & \ddots & \\ & & & c_{N-1} & d_{N-1} \\ & & & & c_0 \end{pmatrix} \begin{pmatrix} \\ \\ \psi \\ \\ \\ \end{pmatrix} \tag{2.13}$$

$$\begin{pmatrix} a_0 & & & \\ b_1 & a_1 & & \\ & \ddots & \ddots & \\ & & b_{N-1} & a_{N-1} \end{pmatrix} \begin{pmatrix} \\ \\ \psi \\ \\ \end{pmatrix} = \begin{pmatrix} c_0 & & & \\ d_1 & c_1 & & \\ & \ddots & \ddots & \\ & & d_{N-1} & c_{N-1} \end{pmatrix} \begin{pmatrix} \\ \\ \phi \\ \\ \end{pmatrix} \tag{2.14}$$

with the coefficients

$$a_j = \hat{S}_1(C_2(j) + e^\varepsilon), \qquad b_j = \hat{C}_1 S_2(j)$$
$$c_j = \hat{C}_1(C_2(j) + e^\varepsilon), \qquad d_j = \hat{S}_1 S_2(j)$$

where $\hat{C}_1 = \cosh K_1^*$, $\hat{S}_1 = \sinh K_1^*$, $C_2(j) = \cosh 2K_2(j)$, $S_2(j) = \sinh 2K_2(j)$, and $K_2(0) = 0$.

These are our basic equations, which will be used in the following sections. Before doing that, however, let us compare them with the formulas of the Hamiltonian limit of large vertical and small horizontal couplings, i.e., $K_1^*, K_2 \ll 1$. Then one can pull the three exponentials in (2.1) together, thus obtaining a single quadratic form of Fermi operators in the exponent. Diagonalizing this form directly leads to simpler equations. In the notation of LSM they read

$$(A + B)\phi = \varepsilon \psi; \qquad (A - B)\psi = \varepsilon \phi \qquad (2.15)$$

Here $(A \pm B)$ are also bidiagonal and the eigenvalue ε only appears on the right-hand sides. These equations also follow from (2.13) and (2.14) if one treats ε, K_1^*, and $K_2(j)$ as small quantities and keeps only first-order terms. This is a useful check for our calculations. At the same time we see how the general case differs from the Hamiltonian limit.

3. SURFACE MAGNETIZATION

In the Ising model the surface magnetization m_s is, from a technical point of view, the simplest order parameter. It can be obtained from the large-distance limit of the spin correlation function in the surface.[8,13] Choosing the left boundary in Fig. 1 and writing the correlation function in terms of W and its spectral representation gives

$$m_s = \langle 1| V_1^{-1/2}\sigma_1^x V_1^{1/2} |0\rangle = \hat{C}_1 \langle 1| \sigma_1^x |0\rangle \qquad (3.1)$$

Here $|0\rangle$ and $|1\rangle$ are the eigenstates of W corresponding to the largest and next to largest eigenvalues, respectively. The result (3.1) strictly holds only in the limit $N \to \infty$, where these two eigenvalues become degenerate. In terms of fermions, $|0\rangle$ is the α vacuum and $|1\rangle$ has the fermion state with the lowest single-particle eigenvalue ε occupied. Denoting this state by p, one has $|1\rangle = \alpha_p^\dagger |0\rangle$ and the quantity ε_p goes to zero as $N \to \infty$ (below the critical temperature).

Using the form of $|0\rangle$ and $|1\rangle$ and expressing $\sigma_1^x = (c_1^\dagger + c_1)$ in terms of the α's, one finds

$$m_s = \hat{C}_1 \cdot \phi_p(1) \qquad (3.2)$$

Thus, one only needs the particular fermion state p to calculate m_s. The nature of this state is known for the homogeneous system[8,13] and for related problems like the XY spin chain.[15] There it is found that for the state p, ϕ_p is localized near the left and ψ_p near the right boundary within a localization length which diverges at the critical point. We now look for the analogous solution in the general inhomogeneous case.

For that purpose we let $N \to \infty$ and look for a solution of (2.13), (2.14) with $\varepsilon = 0$, near the left boundary. With $\varepsilon = 0$ one has $c_0 = 0$ and, from (2.14), $\psi_p(1) = 0$. Using this in (2.13) gives $\psi_p(2) = 0$. In this way one finds that $\psi_p(n) = 0$ for all n, while $\phi_p(n)$ is determined from (2.13) with the right-hand side set equal to zero. The resulting equation can be written as

$$\begin{pmatrix} 1 & \lambda_1 & & \\ & 1 & \lambda_2 & \\ & & \ddots & \ddots \\ & & & \ddots \end{pmatrix} \begin{pmatrix} \\ \phi_p \\ \\ \end{pmatrix} = 0 \tag{3.3}$$

with

$$\lambda_n = \frac{\tanh K_2(n)}{\tanh K_1^*} \tag{3.4}$$

This simple one-step recursion relation for $\phi_p(n)$ is our basic result. For K_1^*, $K_2(n) \ll 1$ the hyperbolic tangents disappear and the Hamiltonian limit of the problem[8] is recovered. Of course, only such solutions of (3.3) are relevant which can be normalized. These appear below the critical temperature, and m_s is then given by

$$m_s = \hat{C}_1 \left[\sum_{n=1}^{\infty} \left(\prod_{i=1}^{n-1} \frac{1}{\lambda_i} \right)^2 \right]^{-1/2} \tag{3.5}$$

This formula determines m_s for arbitrary coupling constants $K_2(n)$.

The simplest case is the homogeneous system with $\lambda_n = \lambda$. Then $\phi_p(n)$ decays exponentially into the interior, $\phi_p(n) \sim \lambda^{-n}$ for $K_2 > K_1^*$, i.e., for $T < T_c$, and one finds the result of McCoy and Wu,[16]

$$m_s = \left(\frac{\cosh 2K_2 - \cosh 2K_1^*}{\cosh 2K_2 - 1} \right)^{1/2} \tag{3.6}$$

This gives the surface critical exponent $\beta_s = 1/2$, independent of the anisotropy. For fixed anisotropy, the value of m_s is larger if the strong couplings are in the surface rather than perpendicular to it.

We now turn to the model of Hilhorst et al. Here the couplings are taken to be $K_2(n) = K_2(1 + an^{-y})$ and thus approach their bulk limit K_2

(which still determines T_c) as a power law. As in the homogeneous case, the function $\phi_p(n)$ will extend a long way into the bulk if one is close to the critical temperature. Its normalization and thus m_s will therefore be determined essentially by values $n \gg 1$. In this region one can expand $\tanh K_2(n)$ to obtain

$$\lambda_n \simeq \lambda(1 + bn^{-y}) \tag{3.7}$$

with

$$b = a \frac{2K_2^c}{\sinh 2K_2^c} \tag{3.8}$$

where K_2^c is the critical value of K_2. The analysis of ref. 8 can then be taken over. In particular, $\phi_p(n)$ varies as $n^{-b}\lambda^{-n}$ for $y = 1$ and large n. This exponential decay modified by a power law leads to a critical exponent

$$\beta_s = \tfrac{1}{2} - b \tag{3.9}$$

which varies continuously with b. Equation (3.9) is the result found by Blöte and Hilhorst[3] in a different way.

Outside the critical region, m_s has to be calculated numerically from (3.5). Results for an isotropic bulk system and $y = 1$ are shown in Fig. 2.
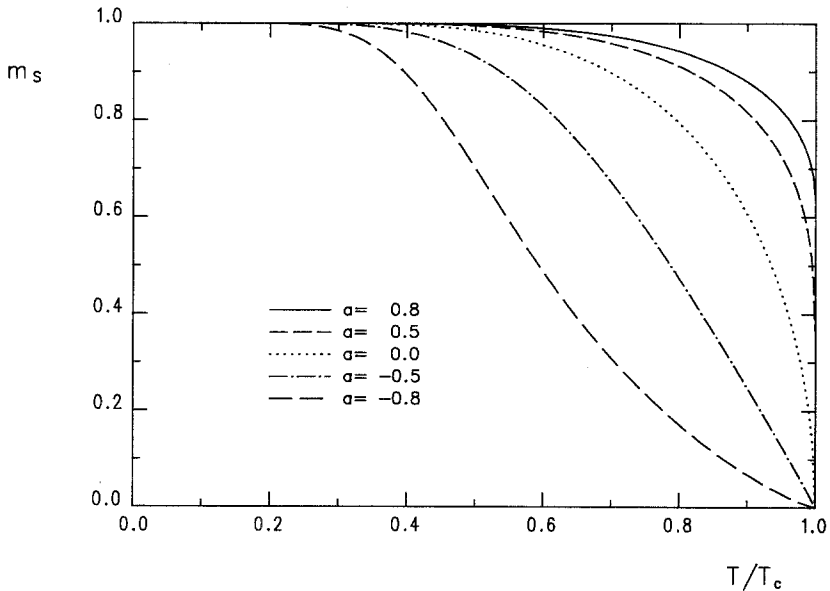

Fig. 2. Surface magnetization m_s in the model of Hilhorst *et al.* for an isotropic bulk system, $y = 1$, and various values of the parameter a; see text.

The curve for $a = 0$ corresponds to the homogeneous system, Eq. (3.6). The general behavior of m_s is easily understood on physical grounds. In particular, m_s is smaller (larger) than in the homogeneous case if the couplings near the surface are reduced (enhanced). The values of the exponent β_s are consistent with this feature. For $b > 1/2$, i.e., if the couplings near the surface are sufficiently enhanced, m_s stays finite as one approaches T_c from below.

With (3.5) one can also investigate the effects of anisotropy in the bulk. In general, one finds the same feature as in the homogeneous system: m_s is larger if the strong couplings are those along the surface. This may be understood by viewing the system as an assembly of chains[17] parallel to the surface. These chains show strong fluctuations and need only a small interchain coupling to become well ordered. The end spins of chains which are perpendicular to the surface generally show less order. An exception occurs only close to T_c and for $a < 0$, as shown in Fig. 3. The reduced couplings $K_2(n)$ near the surface here give a power $\beta_s < 1/2$ if $K_2 \ll K_1$. On the other hand, for $K_2 \gg 1$, $\beta_s = 1/2$, so that m_s rises faster below T_c in this case. Actually, for $K_2 \gg 1$ one has tanh $K_2(n) \simeq 1$ for all bonds except a few ones near the surface. This modification does not change the exponent of the homogeneous system, which explains the value $\beta_s = 1/2$ in this case. In fact, the curve for $K_2/K_1 = 100$ in Fig. 3 can practically be reproduced
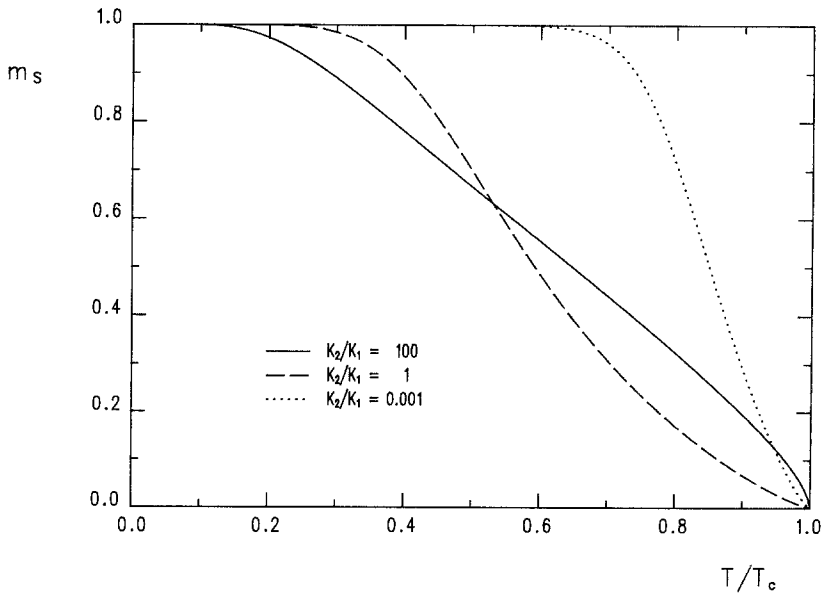

Fig. 3. Surface magnetization m_s in the model of Hilhorst *et al.* for $a = -0.8$, $y = 1$, and three different anisotropies of the bulk system.

by a model where only $K_2(1)$ is different from all the other couplings $K_2(n) = K_2$.

Finally, we note that similar effects will happen in a system where the coupling constants $K_1(n)$ vary with distance from the surface. A calculation with a modified $K_1(1)$ only was done by Au-Yang.[18]

4. CORNER MAGNETIZATION

We now turn to the corners of the lattice in Fig. 1. The magnetization m_c can be obtained from the correlation function of two corner spins in the limit $M, N \to \infty$.[10] Working with the two left corners, this correlation function can be expressed as

$$\Gamma = \frac{\langle B| \, \sigma_1^x V_1^{-1/2} W^M V_1^{-1/2} \sigma_1^x \, |B\rangle}{\langle B| \, V_1^{-1/2} W^M V_1^{-1/2} \, |B\rangle} \tag{4.1}$$

Here the state vector $|B\rangle$ describes the free summation over the boundary spins in the top and bottom rows:

$$|B\rangle = \prod_{n=1}^{N} \frac{1}{\sqrt{2}} (|\sigma_n^x = +1\rangle + |\sigma_n^x = -1\rangle)$$

$$= \prod_{n=1}^{N} |\sigma_n^z = 1\rangle \tag{4.2}$$

Using the form of V_1 and the eigenvectors of W, this leads to

$$m_c = e^{K_1^*} \frac{\langle 1| \, \sigma_1^x \, |B\rangle}{\langle 0|B\rangle} \tag{4.3}$$

which is our starting point. Repeating the steps of ref. 10, one obtains

$$m_c = e^{K_1^*} \left[\phi_p(1) + \sum_q \phi_q(1) f_q \right] \tag{4.4}$$

with

$$f_q = \frac{\langle 1| \, \alpha_q \, |B\rangle}{\langle 0|B\rangle} \tag{4.5}$$

Here p again denotes the surface state and q labels all the other single-fermion eigenstates. The essential point is that the matrix elements f_q can be obtained from the system of linear equations

$$\sum_q f_q h_q(n) = -g_p(n); \qquad n = 1, 2, ..., N \tag{4.6}$$

with g and h introduced in (2.8), and related to ϕ, ψ via (2.12). Thus, one has to find all ϕ and ψ, solve (4.6), and calculate m_c from (4.4).

So far the formulation has been completely general. We now specialize to a homogeneous lattice with $K_2(n) = K_2$. In this case the functions ϕ, ψ can be found analytically, as in the Hamiltonian limit. The algebra, however, is more involved because one has to work with the relations in the hyperbolic triangle.[21] The functions ϕ and ψ can be taken to be real and have the form

$$\phi_q(n) = C_q[\cos(qn) - \beta_q \sin(qn)] \tag{4.7}$$

and similarly for $\psi_q(n)$. Here C_q is a normalization factor and β_q is given explicitly in the Appendix. The inner equations of the linear systems (2.13), (2.14) then lead to the well-known dispersion relation[19]

$$\cosh \varepsilon_q = C_1 C_2 + S_1 S_2 \cos q \tag{4.8}$$

where $S_1 = \sinh 2K_1^*$, $C_1 = \cosh 2K_1^*$. The first and last equations determine the allowed q values and the coefficient β_q. The selection rule for the q values can be written

$$e^{2iqN} = \frac{C_1 S_2 + S_1 C_2 \cos q - iS_1 \sin q}{C_1 S_2 + S_1 C_2 \cos q + iS_1 \sin q}$$
$$= e^{2i\delta(q)} \tag{4.9}$$

which is equivalent to the results of Abraham.[13] The angle $\delta(q)$ [which is related to his quantity $\delta^*(q)$ via $\delta(q) = -\delta^*(\pi - q)$] lies between 0 and $-\pi/2$ for $T < T_c$ and vanishes for $q = 0$ and $q = \pi$. The surface state can be obtained from these relations by the substitution $q \to \pi + ip$. The reflection symmetry of the lattice under $n \to N + 1 - n$ leads to

$$\phi(n) = \pm \psi(N + 1 - n) \tag{4.10}$$

and, therefore, to two classes of eigenstates. As in the Hamiltonian limit, only the states with symmetric functions g and h enter into (4.6). Therefore, Eq. (4.6) only consists of $N/2$ equations for $N/2$ quantities f_q, the other f's being zero.

We have solved (4.6) numerically for finite N and then obtained m_c. The necessary size, for a given accuracy, depends on the temperature and the anisotropy. It increases as one comes closer to T_c. For an isotropic system and $T \leqslant 0.5T_c$, one only needs $N \leqslant 8$ to obtain a four-digit accuracy in m_c. Up to $T = 0.95T_c$, $N \leqslant 120$ is sufficient. If the vertical couplings K_1 become very small, however, the necessary size increases considerably. In a sense, one is then working in the wrong representation. Now, since the cor-

ner magnetization is invariant under the interchange $K_1 \leftrightarrow K_2$, one does not have to treat both cases $K_1 < K_2$ and $K_1 > K_2$. However, it is an important check that the numerical results actually coincide for both cases. Another check was the agreement of the data for an isotropic system with previous results based on a decimation method and the star–triangle transformation.[10]

The results are shown in Fig. 4. First, we note that the critical exponent is $\beta_c = 1$ for all anisotropies. This is what one expects since a rescaling can make the system isotropic without changing the 90° angle at this type of corner[10] and for isotropic systems, $\beta_c = 1$ is well established. Second, m_c increases with the anisotropy, for fixed T/T_c. This may be understood in terms of a picture of coupled chains, as in Section 3. Actually, the same effect appears in the bulk magnetization given by Onsager's formula[20]

$$m_B = (1 - k^2)^{1/8} \tag{4.11}$$

where $k^{-1} = \sinh 2K_1 \cdot \sinh 2K_2$. We see here that the feature exists even at the corners, i.e., the ends of the outermost chains.

One would expect that a simple formula like (4.11) also exists for m_c. In the Hamiltonian limit this is the case. Based on numerical and analytical results the law

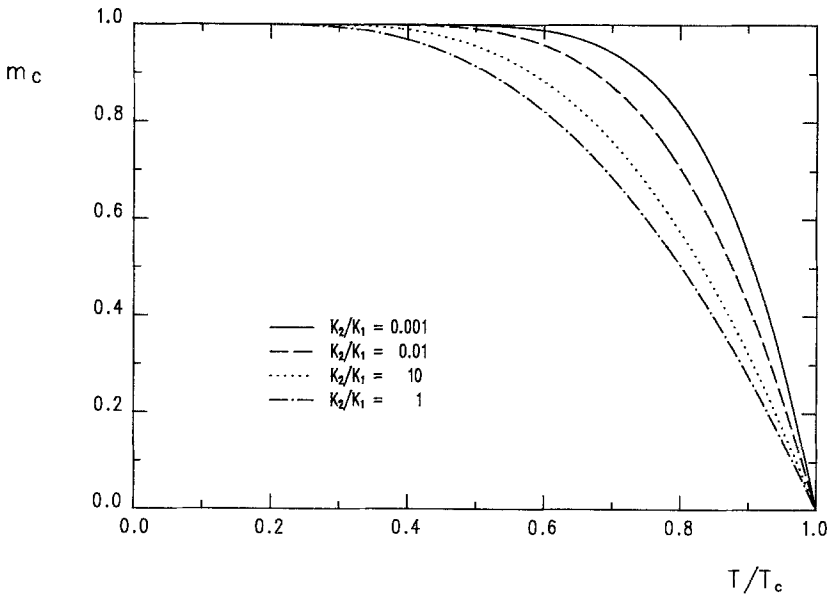$$m_c = 1 - K_1^*/K_2; \qquad K_1^*, K_2 \ll 1 \tag{4.12}$$

Fig. 4. Corner magnetization m_c at a 90° corner of a square lattice for various anisotropies.

was found. We therefore looked for an analytical formula to describe the general case. Indeed, we have found that the simple expression

$$m_c = 1 - \tfrac{1}{2}(\coth K_1 - 1)(\coth K_2 - 1) \tag{4.13}$$

reproduces all the numerical data to at least five decimal places. Moreover, it reduces to (4.12) in the Hamiltonian limit. Therefore, we have no doubts that (4.13) *is* the exact result and represents the corner analogue of Onsager's bulk magnetization formula. An explicit derivation is, of course, still desirable. Note that it is not the Onsager parameter k which enters into the expression for m_c.

4. CONCLUSION

We have studied various boundary effects in an Ising square lattice using the row-to-row transfer matrix. For the surface magnetization we derived and applied a simple, explicit, and general expression. The solution of the corner problem was also given, but here a complete analytical derivation of the final result is still lacking. It could be that a different mathematical technique is needed here. One can also study corners in the square lattice where the edges are along the diagonals of the lattice[12] or corners in a triangular lattice. There the corner magnetization will show even more interesting behavior, but the transfer matrix is more complicated.

APPENDIX

The matrices in Eqs. (2.6), (2.7) are

$$\mathsf{M} = \frac{1}{2} e^{-2K_1^*} \begin{pmatrix} 1 + C_2(1) & -S_2(1) & & & & \\ -S_2(1) & C_2(1) + C_2(2) & -S_2(2) & & & \\ & -S_2(2) & C_2(2) + C_2(3) & & & \\ & & & \ddots & & \\ & & & & & -S_2(N-1) \\ & & & & -S_2(N-1) & 1 + C_2(N-1) \end{pmatrix}$$

$$\mathsf{N} = \frac{1}{2} \begin{pmatrix} 1 - C_2(1) & -S_2(1) & & & \\ S_2(1) & 0 & & & \\ & & \ddots & & \\ & & & 0 & -S_2(N-1) \\ & & & S_2(N-1) & C_2(N-1) - 1 \end{pmatrix}$$

$\hat{\mathsf{M}}$ is obtained from M by changing $\exp(-2K_1^*) \to \exp(2K_1^*)$ and $-S_2(j) \to S_2(j)$. N' denotes the transposed of N.

The coefficient β_q in (4.7) is given by

$$\beta_q = \frac{\cos(q + \delta) - \tanh K_2 \cdot \cos q - \tanh K_1^* \cdot (\tanh K_2 \cdot \cos \delta - 1)}{\sin(q + \delta) - \tanh K_2 \cdot \sin q - \tanh K_1^* \cdot \tanh K_2 \cdot \sin \delta}$$

In the Hamiltonian limit it reduces to

$$\beta_q = \operatorname{ctg}(q + \delta) = \frac{1 + \lambda \cos q}{\lambda \sin q}$$

with $\lambda = K_2/K_1^*$. This corresponds to Eq. (C.2) of ref. 10.

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